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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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And taking V_2:

$$V_2 = \frac{q \sinh q\xi}{\Delta}\left[\sinh \sigma qb \cosh q(a-x) + \alpha \cosh \sigma qb \sinh q(a-x)\right]$$

we have

$$\frac{F(p)}{\Delta'(p)} = \frac{2kq^2 \sinh q\xi\,[\sinh \sigma qb \cosh q(a-x) + \alpha \cosh \sigma qb \sinh q(a-x)]}{(\sigma b + \alpha a)\cosh qa \cosh \sigma qb + (a + \alpha\sigma b)\sinh qa \sinh \sigma qb}$$

so that

$$\frac{1}{\alpha}\,\frac{F(\alpha)}{\Delta'(\alpha)} =$$

$$\frac{1}{-k\lambda^2}\cdot\frac{+2k\lambda^2 \sin \lambda\xi\,[\sin \sigma b\lambda \cos \lambda(a-x) + \alpha \cos \sigma b\lambda \sin \lambda(a-x)]}{(\sigma b + \alpha a)\cos \lambda a \cos \sigma\lambda b + (a + \alpha\sigma b)\sin \lambda a \sin \sigma\lambda b}$$

$$= \frac{-2\sin \lambda\xi\,[\tan \sigma b\lambda(\cos \lambda x + \tan \lambda a \sin \lambda x) + \sigma(\tan \lambda a \cos \lambda x - \sin \lambda x)]}{(\sigma b + \alpha a) - (a + \alpha\sigma b)\tan \lambda a \tan \sigma\lambda b},$$

$$= 2\alpha\,\frac{\sin \lambda\xi \sin \lambda x \sec^2 \lambda a}{(\sigma b + \alpha a) + (a + \alpha\sigma b)\,\alpha \tan^2 a\lambda} \quad \text{on using (6),}$$

and $\dfrac{F(0)}{\Delta(0)} = 0.$

Hence $$V_2 = \sum_{\lambda} \frac{2 \sin \lambda\xi \sin \lambda x\, e^{-k\lambda^2 t}}{a + \sigma/b\alpha\,(\cos^2 \lambda a + \alpha^2 \sin^2 \lambda a)}$$

The summation extends over the positive roots of the equation (6), since negative values of λ do not give any fresh values of p. $\lambda = 0$ is also excluded because $q = 0$ is not a pole of V_2.

It is obvious from the symmetry in x and ξ of the above expression for V_2, that V_1 also reduces to the same value.

Again, to evaluate V_3:

$$V_3 = \frac{q \sinh q\xi \sinh q_1(a + b - x)}{\cosh qa \sinh \sigma qb + \alpha \sinh qa \cosh \sigma qb}$$

Here

$$\frac{F(p)}{\Delta'(p)} = \frac{2kq^2 \sinh q\xi \sinh \sigma q(a + b - x)}{(\sigma b + a\alpha)\cosh qa \cosh \sigma qb + (a + \alpha\sigma b)\sinh qa \sinh \sigma qb}$$

so that

$$\frac{1}{\alpha}\,\frac{F(\alpha)}{\Delta'(\alpha)}=\frac{1}{-kx^2}\cdot\frac{2k\lambda^2\sin\lambda\xi\,\sin\sigma\lambda\,(a+b-x)}{(\sigma b+a\alpha)\cos\lambda a\cos\sigma\lambda b-(a+\alpha\sigma b)\sin\lambda a\sin\sigma\lambda b}$$

$$=\frac{2\sin\lambda\xi\,\sin\sigma\lambda\,(a+b-x)}{(a+\alpha\sigma b)\sin\lambda a\sin\sigma\lambda b-(\sigma\lambda+a\alpha)\cos\lambda a\cos\sigma\lambda b}$$

and $$\frac{F(0)}{\Delta(0)}\doteq 0.$$

Hence

$$V_3=\sum_{\lambda}\frac{2\sin\lambda\xi\,\sin\sigma\lambda\,(a+b-x)\,e^{-k\lambda^2 t}}{(a+\alpha\sigma b)\sin\lambda a\sin\sigma\lambda b-(\sigma b+a\alpha)\cos\lambda a\cos\sigma\lambda b}.$$

Thus the temperature at time t in the two portions of the rod is given by

$$V_{1,2}=\sum_{\lambda}\frac{2\sin\lambda\xi\,\sin\lambda x\,e^{-k\lambda^2 t}}{a+\sigma b/\alpha\,(\cos^2\lambda a+\alpha^2\sin^2\lambda a)}\quad\ldots\quad(7)$$

$$V_3=\sum_{\lambda}\frac{2\sin\lambda\xi\,\sin\sigma\lambda\,(a+b-x)\,e^{-k\lambda^2 t}}{(a+\alpha\sigma b)\sin\lambda a\sin\sigma\lambda b-(\sigma b+a\alpha)\cos\lambda a\cos\sigma\lambda b}.\quad(8)$$

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ON THE QUASI-PERIODIC SOLUTIONS OF MATHIEU'S EQUATIONS AND THE INTEGRAL EQUATIONS CONNECTED WITH THEM.

BY S. C. DHAR, NAGPUR.

Introduction.

In a paper published in the *Journal of the Department of Science*, Calcutta University,* and also in a paper published in the *Journal of the Indian Mathematical Society*†, I obtained integral equations connected with Mathieu Functions. The method I gave there, can be used to obtain integral equations which will give quasi-periodic solutions of Mathieu's Differential equation. Mr. E. G. C. Poole has, in a paper‡ entitled "On Certain Classes of Mathieu Functions", given certain integral equations connected with the quasi-periodic solutions. It is the object of this paper to obtain integral equations as given by Poole by a different and simpler method as used in my paper cited above.

1. The well-known two-dimensional wave-equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial U}{\partial y^2} + k^2 U = 0 \qquad \ldots \quad (1)$$

is transformed by the substitution

$$x + iy = h \cosh (\xi + i\eta)$$

into

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} + h^2 k^2 (\cosh^2 \xi - \cos^2 \eta)\, U = 0 \qquad \ldots \quad (2)$$

If now, we assume particular integrals of the form

$$U = F(\xi) . G(\eta),$$

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* Dhar: *Jour. of Dept. of Science* (Cal Univ.) Vol. III, 1922, pp. 251—256.

† *Jour. of the Indian Math Soc.*, Vol. XVI, 1926.

‡ Poole: *Proc. London Math. Soc.*, Vol. 20., 1922. pp. 374—388.

Also Dhar: *Mathieu Functions*, a book published by the Cal. Univ, in 1928.

we arrive at the following equations:—

$$\left.\begin{array}{ll}\text{(i)} & \dfrac{d^2G}{d\eta^2} + (a - h^2k^2 \cos^2 \eta)\, G = 0 \\ \text{(ii)} & \dfrac{d^2F}{d\xi^2} + (h^2k^2 \cosh^2 \xi - a)\, F = 0,\end{array}\right\} \quad \ldots \quad \ldots \quad (3)$$

where a is an arbitrary constant

We may further transform the equations (3) by the substitution:

$$\lambda = \cosh \xi, \quad \mu = \cos \eta$$

and obtain

$$\left.\begin{array}{l}(1 - \mu^2)\dfrac{d^2G}{d\mu^2} - \mu\dfrac{dG}{d\mu} + (a - h^2k^2\mu^2)\, G = 0 \\ (\lambda^2 - 1)\dfrac{d^2F}{d\lambda^2} + \lambda\dfrac{dF}{d\lambda} + (h^2k^2\lambda^2 - a)\, F = 0\end{array}\right\} \quad \ldots \quad (4)$$

If we take the form of the equation (4) as

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (a - h^2k^2x^2)\, y = 0,$$

we find that this differential equation has regular singularities at $x = \pm 1$, the exponents at these being equal by symmetry. These are found to be 0 and $\frac{1}{2}$. Hence the two fundamental solutions at $x = 1$ can be obtained in the form

$$1 + a_1(1-x) + a_2(1-x)^2 + \ldots + a_n(1-x)^n + \ldots \quad \ldots \quad (\alpha)$$

$$\sqrt{1-x}\,\{1 + b_1(1-x) + b_2(1-x)^2 + \ldots + b_n(1-x)^n + \ldots\} \quad (\beta)$$

Similarly we can write down the solutions at $x = -1$.

From the latter form of the solution (β) it is evident that when $x = \cos \eta$, it can be written in the form

$$\sum_0^\infty a_n \cos\left(n + \frac{1}{2}\right)\eta.$$

Similarly by considering the forms of solutions at $x = -1$, it can be seen that there is a solution of equation (2) (i) of the form

$$\sum_0^\infty b_n \sin\left(n + \frac{1}{2}\right)\eta.$$

The demonstration as regards the existence of these solutions as given by Mr. Poole is interesting. He has shown that for the same value of a, there exist solutions of the form $ce_{n+\frac{1}{2}}(\eta)$ and $se_{n+\frac{1}{2}}(\eta)$ which reduce to $\cos(n+\frac{1}{2})\eta$ and $\sin(n+\frac{1}{2})\eta$ respectively when $h^2k^2 \to 0$, the value of a being $(n+\frac{1}{2})^2$.

2.* If $U(\xi, \eta)$ is an integral of (2), the integral defined by

$$G(\eta) = \int_{-\pi}^{\pi} U(i\theta, \eta)\, \phi(\theta)\, d\theta \qquad \ldots \quad (5)$$

will satisfy the differential equation (3) (i)

when $$\int_{-\pi}^{\pi} \left(\frac{\partial^2}{\partial\eta^2} + a - h^2k^2 \cos^2\eta \right) U(i\theta, \eta)\, \phi(\theta)\, d\theta = 0$$

i.e., if $$\int_{-\pi}^{\pi} \left(\frac{\partial^2 U}{\partial\eta^2} - h^2k^2 \cos^2\eta \,.\, U \right) \phi(\theta)\, d\theta$$

$$+ a \int_{-\pi}^{\pi} U(i\theta, \eta)\, \phi(\theta)\, d\theta = 0.$$

But from (2), we get

$$\frac{\partial^2 U}{\partial\eta^2} - h^2k^2 \cos^2\eta \,.\, U = \frac{\partial^2 U}{\partial\theta^2} - h^2k^2 \cos^2\theta \,.\, U$$

Hence the above integral reduces to

$$\int_{-\pi}^{\pi} \phi(\theta) \left(\frac{\partial^2 U}{\partial\theta^2} - h^2k^2 \cos^2\theta \,.\, U \right) d\theta + a \int_{-\pi}^{\pi} U(i\theta, \eta)\phi(\theta)\, d\theta = 0,$$

which, on integrating by parts, becomes

$$\left[\frac{\partial U}{\partial\theta} \cdot \phi(\theta) - U \frac{\partial\phi}{\partial\theta} \right]_{-\pi}^{\pi}$$

$$+ \int_{-\pi}^{\pi} U(i\theta, \eta) \left[\frac{\partial^2\phi}{\partial\theta^2} + (a - h^2k^2 \cos^2\theta)\, \phi \right] d\theta = 0$$

---

* Dhar: *Mathieu Functions*, Calcutta Univ. Publication, pp. 46-47.

which will be the case, if $\phi(\theta)$ satisfy the equation

$$\frac{\partial^2 \phi}{\partial \theta^2} + (a - h^2 k^2 \cos^2 \theta)\, \phi = 0 \qquad \ldots \qquad (5)$$

and if further

$$\left[\frac{\partial U}{\partial \theta}\, \phi(\theta) - U \frac{\partial \phi}{\partial \theta} \right]_{-\pi}^{\pi} = 0 \qquad \ldots \qquad (6)$$

The first condition implies that ϕ must be a solution of Mathieu's equation as given in (3) (i).

3. We will now proceed to examine the condition (6) fully. This condition will hold if any one of the following sets of conditions be satisfied, *viz.*:—

(i) $U(i\pi, \eta) = U(-i\pi, \eta)$ | $\phi(\pi) = \phi(-\pi)$

$$\left[\frac{\partial U}{\partial \theta}\right]_{\theta=\pi} = \left[\frac{\partial U}{\partial \theta}\right]_{\theta=-\pi} \qquad \left[\frac{\partial \phi}{\partial \theta}\right]_{\pi} = \left[\frac{\partial \phi}{\partial \theta}\right]_{-\pi}$$

This condition was studied by me in my paper already cited to obtain integral equations connected with periodic Mathieu Functions.

If we stipulate that ϕ should be a solution of Mathieu's equation of the quasi-periodic types as discussed in § 1, it will be evident that the condition (6) will be satisfied, if any one of the following sets of conditions be satisfied:

(ii) $U(i\pi, \eta) = 0 = U(-i\pi, \eta)$ | $\phi(\pi) = \phi(-\pi) = 0$

$$\left[\frac{\partial U}{\partial \theta}\right]_{\pi} = -\left[\frac{\partial U}{\partial \theta}\right]_{-\pi} \qquad \left[\frac{\partial \phi}{\partial \theta}\right]_{\pi} = -\left[\frac{\partial \phi}{\partial \theta}\right]_{-\pi}$$

and

(iii) $U(i\pi, \eta) = -U(-i\pi, \eta)$ | $\phi(\pi) = -\phi(-\pi)$

$$\left[\frac{\partial U}{\partial \theta}\right]_{\pi} = 0 = \left[\frac{\partial U}{\partial \theta}\right]_{-\pi} \qquad \left[\frac{\partial \phi}{\partial \theta}\right]_{\pi} = 0 = \left[\frac{\partial \phi}{\partial \theta}\right]_{-\pi}$$

The above two conditions are the only two that are relevant to the present problem.

4. From the set of conditions as given by (ii), it is evident that if $\phi(\theta)$ be a quasi-periodic solution of the form $ce_{n+\frac{1}{2}}(\theta)$, then the conditions imposed on ϕ will be satisfied.

Now, if we examine the conditions imposed on U in (ii), it is evident that since U is to be a solution of (2) and therefore symmetrical in θ and η, it can in particular be expressible as product of two functions of the form

$$U(i\theta, \eta) = F_{2m+1}(\cos \tfrac{1}{2}\theta \cos \tfrac{1}{2}\eta)\, F'_{2m}(\sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}\eta)$$

where F_{2m+1} is a function which can be expanded in a series of *odd* positive integral powers of the argument, and F'_{2m} in a series of *even* positive integral powers. $U(i\theta, \eta)$ is thus an even function in θ as well as in η, inasmuch as when θ is changed to $-\theta$, and η into $-\eta$, the function $U(i\theta, \eta)$ does not alter.

Hence the equation (5) takes up the form :—

$$ce_{n+\frac{1}{2}}(\eta) =$$

$$\lambda \int_{-\pi}^{\pi} F_{2m+1}(\cos \tfrac{1}{2}\theta \cos \tfrac{1}{2}\eta)\, F'_{2m}(\sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}\eta)\, ce_{n+\frac{1}{2}}(\theta)\, d\theta \quad \dots (6)$$

The particular forms which F and F′ must assume can now be easily obtained from the fact that $U = F \cdot F'$ and must satisfy equation (2).

Proceeding in the same way with respect to the conditions (iii) we will arrive at the following integral equation

$$se_{n+\frac{1}{2}}(\eta) =$$

$$\mu \int_{-\pi}^{\pi} F_{2m}(\cos \tfrac{1}{2}\theta \cos \tfrac{1}{2}\eta)\, F'_{2m+1}(\sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}\eta)\, se_{n+\frac{1}{2}}(\theta)\, d\theta \quad \dots (7)$$

5. To obtain the forms of F and F′, we will transform the equation (2) by the substitution, which suggests itself from the foregoing arguments, *viz.*

$$\alpha = 2 \cosh \tfrac{1}{2}\xi \cos \tfrac{1}{2}\eta, \quad \beta = 2 \sinh \tfrac{1}{2}\xi \sin \tfrac{1}{2}\eta$$

The equation (2) reduces itself to

$$\frac{\partial^2 U}{\partial \alpha^2} + \frac{\partial^2 U}{\partial \beta^2} + h^2 k^2 (\alpha^2 + \beta^2)\, U = 0 \quad \dots (8)$$

If we now take $U = FF'$, where F is a function of α only, and F′ of β, then the equation (3) resolves itself into the two following

equations:—

$$\frac{d^2F}{d\alpha^2} + (k^2h^2\alpha^2 + A)\,F = 0 \qquad \dots (9)$$

$$\frac{d^2F'}{d\beta^2} + (k^2h^2\beta^2 - A)\,F' = 0 \qquad \dots (10)$$

where A is any arbitrary constant.

It is easy now to obtain a form of the solution F which will involve α in even powers. For following Mr. Poole if we take $A = -ikh$, then

$$F = e^{\frac{1}{2}ikh\alpha^2}$$

and then $F' = e^{-\frac{1}{2}ikh\beta^2}$.

Now from the even form of F, we can easily obtain the odd form as

$$F = e^{\frac{1}{2}ikh\alpha^2} \int_0^{\alpha} e^{-ikht^2}\,dt.$$

Thus the kernel of the integral equation (6) will be:—

$$U(i\theta, \eta) \sim e^{ikh\cos\theta\cos\eta} \int_0^{2\cos\frac{1}{2}\theta\cos\frac{1}{2}\eta} e^{-ikht^2}\,dt. \qquad (11)$$

Similarly the kernel of the integral equation (7) will be

$$U(i\theta, \eta) \sim e^{ikh\cos\theta\cos\eta} \int_0^{2\sin\frac{1}{2}\theta\sin\frac{1}{2}\eta} e^{-ikht^2}\,dt. \qquad (12)$$

I hope to resume discussion of it on some future occasion.

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Nagpur.

THE SYZYGETIC PENCIL OF CUBICS

By S. Audinarayanan.

All order cubics through any given eight points have a ninth point in common and constitute a pencil determined by any two cubics through the eight points. It is known that a cubic and its Hessian intersect in the nine inflexional points of the cubic, which are also the inflexional points of the Hessian. The pencil determined by these two cubics is called a syzygetic pencil. Referred to an inflexional triangle, the equations of the cubics of the pencil will be

$$x^3 + y^3 + z^3 + 6kxyz = 0$$

for all values of k, and the nine points will be inflexional points for each of the cubics. H. S. White has shown that to every order cubic Γ, there exists a unique class cubic Θ, doubly apolar to it, namely, every first polar of Γ being apolar to every first polar of Θ. All the doubly apolar cubics of the cubics of the syzygetic pencil form a syzygetic range given by the equation:

$$l^3 + m^3 + n^3 + 6k'lmn = 0$$

where $2kk' + 1 = 0$. These have the nine base lines for cuspidal tangents.

It is easily seen that the Hessian of Γ is the Cayleyan of Θ, and the Cayleyan of Γ is the Hessian of Θ. This duality persists between the properties of the two curves and of the conics associated with them. Hence if we consider the properties of the syzygetic pencil of cubics, the dual case of the syzygetic range of cubics can be easily derived.

Let us consider the cubics of the syzygetic pencil

$$x^3 + y^3 + z^3 + 6kxyz = 0$$

as constituting a binary domain, and represent each by its parameter k. There are in this pencil two groups of four special members, for there are four degenerate cubics corresponding to the parameters:

$$\left(\infty, -\frac{1}{2}, -\frac{\omega}{2}, -\frac{\omega^2}{2}\right)$$

where $\omega^3 = 1$, while the non-degenerate cubics of the pencil that have these degenerate cubics for Hessians are four in number with parameters $(0, 1, \omega, \omega^2)$. These cubics are termed equi-anharmonic since the pencil of tangent lines from any point on it to itself is equi-anharmonic. The parameters of these two tetrads of cubics are the roots of the two quartics $8x^3 + 1$ and $x^4 - x$ respectively. These quartics which are self-apolar and each the Hessian of the other have been called "conjugate self-apolar quartics" by Dr. R. Vaidyanathaswamy.* It has also been established by him that each of the two quartics determines a null pencil of binary cubics whose Jacobian quartic is the other, and

THEOREM 1. — *If $(x - \alpha)(x - \beta)(x - \gamma)$ belongs to a null pencil, then $(x - \beta)(x - \gamma)$ is the second polar of α with respect to the Jacobian quartic of the pencil.*

Now if $(\alpha\beta\gamma)$ is a member of the null pencil whose Jacobian quartic is $8x^3 + 1$, the covariant relation between the three cubics α, β, γ in the syzygetic pencil can be easily determined. By Theorem (1), given α, β and γ are the roots of the quadratic

$$4\alpha x^2 + 4\alpha^2 x + 1 = 0.$$

Hence
$$4\alpha\beta^2 + 4\alpha^2\beta + 1 = 0$$

or
$$4(\alpha^2 - \beta^2) + \frac{1}{\beta} - \frac{1}{\alpha} = 0$$

or
$$\frac{4\alpha^3 - 1}{\alpha} = \frac{4\beta^3 - 1}{\beta} = \frac{4\gamma^3 - 1}{\gamma}.$$

Thus the three cubics have the same Cayleyan cubic and hence the triads of a null pencil whose Jacobian is $8x^3 + 1$ are the parameters of co-Cayleyan cubics in the syzygetic pencil. We shall call the quartic $8x^3+1$ the "co-Cayleyan quartic" and the null pencil, the "co-Cayleyan null pencil," and $(\beta\gamma)$ the "co-Cayleyan associates" of α. Also if $(\alpha'\beta'\gamma')$ be a member of the associated null pencil whose Jacobian quartic is $x^4 - x$, then for any α', β' and γ' are the roots of the quadratic

$$2\alpha'^2 x^2 - \alpha' - x = 0$$

* Cf. the Pamphlet on "The linear system of binary cubics and quartics,"

and hence satisfy the relation

$$-\frac{1+2\alpha'^3}{6\alpha'^2} = -\frac{1+2\beta'^3}{6\beta'^2} = -\frac{1+2\gamma'^3}{6\gamma'^2}$$

This shows that $(\alpha'\beta'\gamma')$ are co-Hessian cubics of the syzygetic pencil. We shall call this associated null pencil, the "co-Hessian null pencil" the quartic $x^4 - x$, the "co-Hessian quartic", and β', γ', "co-Hessian associates of α'." Thus:

> Two null pencils of binary cubics N_1, N_2 whose Jacobian quartics J_1, J_2 are conjugate self-apolar quartics, correspond to the co-Cayleyan and the co-Hessian triads of cubics in the syzygetic pencil of cubics, and the roots of J_1 and J_2 correspond to the four degenerate cubics and the four equi-anharmonic cubics respectively of the syzygetic pencil.

Now, let us consider two cubics λ, μ of the pencil. The co-Cayleyan associate pairs of λ, μ are the roots of the two quadratics

$$4\lambda x^2 + 4\lambda^2 x + 1 = 0$$
$$4\mu x^2 + 4\mu^2 x + 1 = 0.$$

The Jacobian quadratic $4\lambda\mu x^2 - (\lambda + \mu + 2x)$ is the mixed polar of λ, μ with respect to the co-Hessian quartic. Similarly the Jacobian quadratic of two quadratics whose roots are the co-Hessian associates of λ and μ is the mixed polar of λ and μ with respect to the co-Cayleyan quartic. Let the roots of the two Jacobian quadratics:

$$4\lambda\mu x^2 - (\lambda + \mu + 2x) = 0$$
$$2(\lambda + \mu)x^2 + 4\lambda\mu x + 1 = 0$$

be (α, β) and (γ, δ) respectively. Then if we write $(\alpha\beta)$, $(\lambda\mu)$, $(\gamma\delta)$ in cyclic order the relations of $(\lambda\mu)$ and $(\alpha\beta)$ to $(\gamma\delta)$ and $(\gamma\delta)$ $(\lambda\mu)$ to $(\alpha\beta)$ are the same as of $(\alpha\beta)$ and $(\gamma\delta)$ to $(\lambda\mu)$. Thus the three pairs form a closed set symmetrically related.

Prof. H. S. White* has found some characteristic properties of these co-Hessian and co-Cayleyan triads of cubics and conics related to them. Before coming to these properties we shall define a poloconic of an order conic S with respect to an order cubic Γ as the locus of points whose polar conics with respect to Γ are in-polar to S. When S is

* *Transactions of the American Mathematical Society.* Vol. I, p. 1.

identical with its poloconic it is termed an autopoloconic of the cubic. Also, the envelope of lines whose polo-conics are out-polar to S is a class conic. When this class-conic is identical with S, it is termed a conic auto-polar of the cubic Γ. Now White's result is that

> The co-Cayleyan and the co-Hessian triads of cubics in the syzygetic pencil are characterised by the property that the polar-conics of any one are auto-poloconics of both the others of a co-Cayleyan triad, and conic auto-polars of the two doubly apolar cubics of the other two of the co-Hessian triad.

Thus for every cubic Γ of the syzygetic pencil there exist two pairs of cubics (A, B), (α, β) in the pencil such that the polar-conics of Γ are auto-poloconics of A and B and conic auto-polars of the doubly apolar cubics of α and β. Also the conic polars of the curves A, B, α, β are auto-poloconics respectively to the curves, B, A, β, α and conic auto-polars respectively to the curves β, α, B, A.

The rest of the paper will concern with the extensions of these ideas to any two cubics of the syzygetic pencil. To this end, I define a *bi-poloconic* of an order conic S with respect to two order cubics λ, μ as the locus of a point P, the ϕ-conic of whose polar conics with respect to the cubics λ and μ is in-polar to the given order conic S. When λ and μ denote the same cubic, the bi-poloconic of S is simply the poloconic of S, with respect to that cubic. When S is identical with its bi-poloconic, it is termed an auto-bi-poloconic. Also, I define the bi-conic-autopolar of two cubics λ and μ as the conic S that is identical with the envelope of lines whose bi-poloconics with respect to λ and μ are out-polar to S.

It will be noticed that

> The auto-bi-poloconics of two cubics λ and μ are the polar-conics of two other cubics (γ, δ), where (γ, δ) are roots of the mixed polar of (λ, μ) with respect to the co-Cayleyan quartic.

For, if

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

the bi-poloconic of S with respect to λ and μ is

$$x^2 (a\lambda\mu + f\overline{\lambda + \mu}) + y^2 (b\lambda\mu + g\overline{\lambda + \mu}) + z^2 (c\lambda\mu + h\overline{\lambda + \mu})$$
$$- yz(a + 2f\lambda\mu) - zx(b + 2g\lambda\mu) - xy(c + 2h\lambda\mu) = 0.$$

The defining equations of the auto-bipoloconic fall into three pairs of equations of condition like

$$a\lambda\mu + f\overline{\lambda + \mu} = ka$$

$$\frac{a}{2} + \lambda\mu f = -kf.$$

setting $f/a = x$ and eliminating k we have

$$2x^2(\lambda + \mu) + 4x\lambda\mu + 1 = 0.$$

If γ and δ are the roots of the equation the corresponding nets of conics are

$$x_1x^2 + y_1y^2 + z_1z^2 + 2\gamma(x_1yz + xy_1z + xyz_1) = 0,$$

and $$x_1x^2 + y_1y^2 + z_1z^2 + 2\delta(x_1yz + xy_1z + xyz_1) = 0,$$

$x, y, z,$ being arbitrary. These are the nets of conic-polars of the cubics

$$x^3 + y^3 + z^3 + 6\gamma xyz = 0,$$

and $$x^3 + y^3 + z^3 + 6\delta xyz = 0,$$

which belong to the same syzygetic pencil and (γ, δ) are the roots of the mixed polar ef (λ, μ) with respect to the co-Cayleyan quartic.

Secondly the defining equations of condition for a conic

$$S = al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0$$

to be a bi-conic-auto-polar with respect to the cubics λ and μ fall into three pairs of equations like

$$a\lambda\mu - f = ka$$

$$a(\lambda + \mu) - 2f\lambda\mu = 2kf$$

Setting $f/a = x$ and eliminating k we have

$$2x^2 - 4\lambda\mu x + \lambda + \mu = 0.$$

The two roots of this quadratic (α, β) give two webs of conics which are polo-conics of the two class cubics

$$l^3 + m^3 + n^3 + 6\alpha\, lmn = 0,$$

and $$l^3 + m^3 + n^3 + 6\beta\, lmn = 0.$$

These can be easily seen to be defined in the syzygetic range of cubics determined by the doubly apolar-cubics (λ', μ') of λ and μ where (α, β) are the roots of the mixed polar of $(\lambda'\mu')$ with respect to the co-Hessian

quartic. Also, the two cubics α and β are doubly apolar to the two order cubics γ and δ.

Thus given two cubics (λ, μ) of a syzygetic pencil of cubics, there exists two pairs of cubics (α, β) and (γ, δ) in the same pencil such that the polar-conics of (λ, μ) are auto-bi-polo-conics of (α, β) and bi-conic auto-polars of the two doubly apolar cubics of (γ, δ), where (α, β), (γ, δ) are the roots of the mixed polars of (λ, μ) with respect to the co-Hessian and the co-Cayleyan quartics. Also the three pairs of cubics form a closed system.

Now if we take any three cubics (λ, μ, ν) of the syzygetic pencil, it can be easily verified that

The locus of points whose polar conics with respect to the three cubics $\lambda \mu \nu$ are such that one is out-polar to the ϕ-conic of the other two is a cubic (σ) of the same pencil and the tetrad $(\lambda \mu \nu \sigma)$ is apolar to the co-Cayleyan quartic. The cubics are symmetric with respect to this property.

Particular cases

When $\lambda \mu \nu$ represent the same cubic we find that σ is the Hessian of that cubic.

Secondly, if σ should coincide with one of $\lambda \mu \nu$, say ν, then that cubic ν is one of the cubics whose polar conics are auto-bi-polo-conics of the other two.

Thirdly, if λ and μ should coincide, then the cubic σ and ν separates harmonically the two co-Cayleyan cubics of λ. For, the relation between λ, ν, δ is

$$2\lambda^2(\nu + \sigma) + 4\lambda\nu\sigma + 1 = 0$$

which is exactly the condition that the two quadratics

$$4\lambda x^2 + 4\lambda^2 x + 1.$$

and $$x^2 - \overline{\nu + \sigma}\, x + \nu\sigma$$ be apolar.

The four cubics $(\lambda \mu \nu \sigma)$ possess a very interesting property, namely:

Let P be a point on the cubic λ and be considered as a class-conic. The envelope of lines whose bi-polo conics with respect to (μ, ν) are out-polar to P is a class-conic which is apolar to the polar-conic of P with respect to the cubic σ. This property is symmetric with respect to the cubics $(\lambda \mu \nu \sigma)$.

ON THE GREATEST PRIME FACTOR OF A CERTAIN PRODUCT

By S. D. Chowla.

1. Let P_x denote the greatest prime factor of the product

$$f(x) = (1 + 1^2)(1 + 2^2)\ldots\ldots(1 + x^2).$$

Then, it is known that*

$$\lim_{x \to \infty} \frac{P_x}{x} = \infty.$$

In this note I obtain the improved result:

If $\varepsilon > 0$, then we can find a number ξ such that

$$P_x > (2 - \varepsilon)\, x \log x$$

for all integers $x > \xi = \xi(\varepsilon)$.

2. If possible, suppose that $f(x)$ has no prime factor exceeding $(2 - \varepsilon)\, x \log x$. It would follow that

$$\log f(x) = \sum_p \sum_{m=1}^{\infty} \{\theta(p^m) . \log p\} \qquad \ldots \quad (1)$$

where p runs through all primes $(2, 3, 5 \ldots \ldots)$ not exceeding $(2 - \varepsilon)\, x \log x$, and $\theta(n)$ is the number of numbers y for which

$$y^2 + 1 \equiv 0 \pmod{n}$$

where $y = 1, 2, 3, \ldots \ldots, x$.

The second member of (1) is equal to the sum of the following expressions:

$$\phi(x) = \sum_s \sum_{m=1}^{\infty} \{\theta(s^m) . \log s\} \qquad \ldots \quad (2)$$

where s runs through all primes for which $2 \leqslant s < 2x$, and

$$\psi(x) = \sum_t \sum_{m=1}^{\infty} \{\theta(t^m) . \log t\}, \qquad \ldots \quad (3)$$

where t runs through all primes for which $2x < t \leqslant (2 - \varepsilon)\, x \log x$.

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* See Landau's "*Handbuch der Lehre von der Verteilung der Primzahlen,*" pages 559—561 for a proof.

3. In this section we consider the sum (2). It is easy to see that

$$\theta(2) \leqslant \frac{x+1}{2},$$

and also that $\theta(2^m) = 0$, if $m > 1$.

Moreover, $\theta(p^m) = 0$, if $p \equiv 3 \pmod{4}$,

and $\theta(p^m) \leqslant 2\left(\frac{x}{p^m} + 1\right)$, if $p \equiv 1 \pmod{4}$.

From these inequalities it follows that

$$\phi(x) \leqslant \log 2 . \frac{x+1}{2}$$

$$+ \sum_{\substack{p \equiv 1 \,(\mathrm{mod.}\, 4) \\ p < 2x}} \log p \left\{ \left(\frac{2x}{p} + 2\right) + \left(\frac{2x}{p^2} + 2\right) + \dots + \left(\frac{2x}{p^m} + 2\right) \right\}. \qquad \dots (4)$$

where m is the greatest integer for which $p^m \leqslant 1 + x^2$, and

hence $$m \leqslant \frac{\log(1+x^2)}{\log p}.$$

From (4) it follows that

$$\phi(x) \leqslant O(x) + 2x . \sum_{\substack{p \equiv 1 \,(\mathrm{mod.}\, 4) \\ p < 2x}} \frac{\log p}{p-1} + O \sum_{\substack{p \equiv 1 \,(\mathrm{mod.}\, 4) \\ p < 2x}} \left(\log p . \frac{\log x}{\log p}\right)$$

$$= O(x) + 2x \sum_{\substack{p \equiv 1 \,(\mathrm{mod.}\, 4) \\ p < 2x}} \frac{\log p}{p} + O\left(\log x . \frac{x}{\log x}\right)$$

$$= x \log x + O(x), \qquad \dots \qquad \dots (5)$$

since*

$$\sum_{\substack{p \equiv 1 \,(\mathrm{mod}\, 4) \\ p \leqslant x}} \frac{\log p}{p} = \tfrac{1}{2} \log x + O(1).$$

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* See page 450 of the book cited.

4. In this section we consider the sum (3).

If p is a prime greater than $2x$, then obviously

$$\theta(p^m) = 0, \text{ if } m > 1,$$

and
$$\theta(p) = 0, \text{ if } p \equiv 3 \pmod{4},$$

and
$$\theta(p) \leqslant 1, \text{ if } p \equiv 1 \pmod{4}.$$

Hence

$$\psi(x) \leqslant \sum_{\substack{p \equiv 1 \pmod{4} \\ 2x < p \leqslant (2-\varepsilon) x \log x}} (\log p)$$

$$\leqslant \sum_{\substack{p \equiv 1 \pmod{4} \\ 1 < p \leqslant (2-\varepsilon) x \log x}} (\log p)$$

$$= \left(1 - \frac{\varepsilon}{2}\right) x \log x + o(x \log x) \quad \ldots \quad (6)$$

using the classical result on page 468 of Landau's book.

5. From (1), (2), (3), (5) and (6) it follows that

$$\log f(x) \leqslant \left(2 - \frac{\varepsilon}{2}\right) x \log x + o(x \log x) \quad \ldots \quad (7)$$

Also from the definition of $f(x)$ it follows that

$$\log f(x) = 2x \log x + O(x) \quad \ldots \quad (8)$$

Since ε is positive, (7) and (8) contradict each other. Hence our assumption that $f(x)$ has no prime factor exceeding $(2 - \varepsilon) x \log x$, was wrong.

AN ORDER RESULT INVOLVING EULER'S ϕ-FUNCTION.

By S. D. Chowla.

§ 1. It is well known that

$$\sum_{1}^{X} \phi(n) = \frac{3}{\pi^2} X^2 + O(X \log X)$$

where $\phi(n)$ is the number of positive integers less than and prime to n. In this note I prove the following generalization :—

$$\sum_{1}^{X} \phi^r(n) = A_r X^{r+1} + O(X^r \log^r X) \qquad \dots \quad (1)$$

where r is any positive integer, and

$$A_r = \frac{1}{(r+1)} \prod_p \left\{ 1 - \frac{{}^rC_1}{p^2} + \frac{{}^rC_2}{p^3} - + \dots + \frac{(-1)^r}{p^{r+1}} \right\};$$

and p runs through all primes.

§ 2. We have (if the real part of s is greater than 1)

$$\sum_{n=1}^{\infty} \left\{ \frac{\phi(n)}{n} \right\}^r \frac{1}{n^s} = \prod_p \left\{ 1 + \sum_{m=1}^{\infty} \frac{(1-p^{-1})^r}{p^{ms}} \right\} \qquad \dots \quad (2)$$

where p runs through all primes.

Now

$$(1-p^{-s}) \times \left\{ 1 + \sum_{m=1}^{\infty} \frac{(1-p^{-1})^r}{p^{ms}} \right\}$$

$$= 1 - \frac{{}^rC_1}{p^{s+1}} + \frac{{}^rC_2}{p^{s+2}} - + \dots + \frac{(-1)^r}{p^{s+r}} \qquad \dots \quad (3)$$

and

$$\frac{1}{\zeta(s)} = \prod_p (1-p^{-s}) \qquad \dots \quad (4)$$

From (2), (3) and (4) we have

$$\sum_{n=1}^{\infty} \left\{ \frac{\phi(n)}{n} \right\}^r \frac{1}{n^s} = \zeta(s) \prod_p \left\{ 1 - \frac{{}^rC_1}{p^{s+1}} + \frac{{}^rC_2}{p^{s+2}} - + \dots + \frac{(-1)^r}{p^{s+r}} \right\}$$

$$= \zeta(s) . \sum^{\infty} \frac{a_n}{n^s} \quad \dots \quad \dots \quad (5)$$

The series $\sum_{1}^{\infty} \frac{a_n}{n^s}$ is absolutely convergent if the real part of s is positive.

From (5)

$$\left\{ \frac{\phi(n)}{n} \right\}^r = \sum_{d/n} a_d .$$

Hence

$$\sum_{n=1}^{X} \left\{ \frac{\phi(n)}{n} \right\}^r = \sum_{n=1}^{X} a_n \left[\frac{X}{n} \right] \quad \dots \quad (6)$$

where $[\theta]$ denotes the greatest integer contained in θ.

From (6) we derive

$$\sum_{1}^{X} \left\{ \frac{\phi(n)}{n} \right\}^r = X \left\{ \sum_{1}^{X} \frac{a_n}{n} \right\} + E(X), \quad \dots \quad (7)$$

where

$$E(X) < \sum_{1}^{X} | a_n | . \quad \dots \quad (8)$$

Now

$$\sum_{1}^{\infty} \frac{n a_n}{n^s} = \prod_p \left\{ 1 - \frac{{}^rC_1}{p^s} + \frac{{}^rC_2}{p^{s+1}} - + \dots + \frac{(-1)^r}{p^{s+r-1}} \right\}.$$

Hence

$$| n a_n | \leqslant b_n, \quad \dots \quad (9)$$

where

$$\sum_{1}^{\infty} \frac{b_n}{n^s} = \prod_{p} \left\{ 1 + \frac{{}^rC_1}{p^s} + \frac{{}^rC_2}{p^{s+1}} + \dots + \frac{1}{p^{s+r-1}} \right\}$$

$$= \{ \zeta(s) \}^r \, F(s),$$

where F (s) is absolutely convergent if the real part of s is positive.

Hence

$$\lim_{X \to \infty} \frac{\sum_{1}^{X} b_n}{X \log^{r-1}(X)} = F(1) \qquad \dots \qquad \dots \quad (10)$$

From (9) and (10) we get

$$\sum_{1}^{X} n \mid a_n \mid = O\,(X \log^{r-1} X)$$

whence

$$\sum_{1}^{X} \mid a_n \mid = O\,(\log^r X) \qquad \dots \qquad \dots \quad (11)$$

From (7), (8) and (11) we get

$$\sum_{1}^{X} \left\{ \frac{\phi(n)}{n} \right\}^r = X \left\{ \sum_{1}^{\infty} \frac{a_n}{n} \right\} + O\,(\log^r X) - X \sum_{X+1}^{\infty} \frac{a_n}{n} \dots \quad (12)$$

Now

$$\left| \sum_{X+1}^{\infty} \frac{a_n}{n} \right| < \text{or} = \sum_{X+1}^{\infty} \frac{b_n}{n^2}$$

Let $S_n = b_1 + b_2 + \dots + b_n$; then

$$\sum_{X+1}^{\infty} \frac{b_n}{n^2} = \sum_{X+1}^{\infty} \frac{S_n - S_{n-1}}{n^2}$$

$$= \frac{(-S_X)}{(X+1)^2} + \sum_{X+1}^{\infty} S_n \left\{ \frac{1}{n^2} - \frac{1}{(n+1)^2} \right\}$$

$$= O\left(\frac{\log^{r-1} X}{X}\right) + O\left(\sum_{X}^{\infty} \frac{S_n}{n^3}\right)$$

$$= O\left(\frac{\log^{r-1} X}{X}\right) + O\left(\sum_{X}^{\infty} \frac{\log^{r-1}(n)}{n^2}\right)$$

$$= O\left(\frac{\log^{r-1} X}{X}\right) \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

if r is an integer $>$ or equal to 1.

From (12), (13) and (14) we get

$$\sum_{1}^{X} \left\{\frac{\phi(n)}{n}\right\}^r = X\left(\sum_{1}^{\infty} \frac{a_n}{n}\right) + O(\log^r X) \quad \ldots \quad (15)$$

Now if

$$c_1 + c_2 + \ldots + c_x = Ax + O(\log^r x)$$

it is easy to deduce that

$$1^r c_1 + 2^r c_2 + \ldots + x^r c_x = \frac{Ax^{r+1}}{(r+1)} + O(x^r \log^r x)$$

From this fact and (15) it follows that

$$\sum_{1}^{X} \phi^r(n) = B X^{r+1} + O(X^r \log^r X)$$

where B is a certain constant.

ON THE ORDER OF N (R), THE NUMBER OF TERMS IN THE PERIOD OF THE CONTINUED FRACTION FOR $\sqrt{R}$.

By S. D. Chowla.

1. Let us denote by N (R) the number of terms in the period of the simple continued fraction for $\sqrt{R}$, where R is a positive integer. Dr. T. Vijayaraghavan has recently* proved that

$$N(R) < K . \sqrt{R} . \log R$$

where K is a definite constant. The best value that we can obtain for K by his argument (see p. 411 of the paper) seems to be

$$\frac{1}{\log^e \left(\frac{\sqrt{5}+1}{2}\right)} + \varepsilon,$$

where $\varepsilon > 0$ is arbitrarily small, and R is greater than some value depending on ε.

Let R be *quadratfrei*, and let $\varepsilon > 0$ be arbitrarily small; then I here prove the better result:

$$N(R) < \left(\frac{6}{\pi^2} + \varepsilon\right) \sqrt{R} . \log R$$

provided R (quadratfrei) exceeds a certain limit depending on ε.

2. We need the following simple lemma which is easily proved:

Lemma. *If R is quadratfrei, then whether R is prime to m or not, the congruence*

$$R - x^2 \equiv 0 \pmod{m}$$

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* *Proc. Lond. Math. Soc.* (1927), 403–414.

cannot have more than $2^{\varpi(m)}$ ***incongruent solutions (mod. m), where*** $\varpi(m)$ ***is the number of distinct prime factors of m.***

The case when R is prime to m is well known*.

3. Following the notation of Vijayaraghavan's paper, we write

$$\sqrt{R} = [a_1, a_2, \ldots, a_n, \theta_n]$$

where

$$\theta_n = \frac{P_n + \sqrt{R}}{Q_n},$$

and if $n = 1, 2, 3, 4, \ldots$, we have

$$0 < P_n < \sqrt{R};\ \sqrt{R} - P_n < Q_n < \sqrt{R} + P_n;\ Q_n \mid (R - P_n^2).$$

From these conditions it follows that

$$N(R) \leqslant \sum_{x=1}^{\sqrt{R}} f(R - x^2)$$

where $f(R - x^2)$ denotes the number of divisors of $R - x^2$ which lie between $\sqrt{R} - x$ and $\sqrt{R} + x$. Hence

$$N(R) \leqslant 2 \sum_{x=1}^{\sqrt{R}} \phi(R - x^2)$$

where $\phi(R - x^2)$ is the number of divisors of $R - x^2$ which lie between $\sqrt{R} - x$ and $\sqrt{R}$ Let m be such a divisor; then

$$\sqrt{R} - x < m < \sqrt{R}$$

and hence

$$\sqrt{R} - m < x < \sqrt{R}.$$

It follows that

$$\sum_{1}^{\sqrt{R}} \phi(R - x^2) = \sum_{1}^{\sqrt{R}} \theta(m)$$

where $\theta(m)$ is the number of solutions of the congruence

$$R - x^2 \equiv 0 \pmod{m}$$

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* See *Matthew's Theory of Numbers*, § 35, p. 36
or Landau's *Vorlesungen über Zahlentheorie*, Satz 88, p. 47, Volume I.

where x is subject to the inequality

$$\sqrt{R} - m < x < \sqrt{R}.$$

Since x is restricted by the last inequality to have only m consecutive values, it follows that $\theta(m) \leqslant 2^{\varpi(m)}$, by using the lemma of § 2. Hence

$$N(R) \leqslant 2 \sum_{1}^{\sqrt{R}} \theta(m),$$

$$\leqslant 2\left\{ \sum_{1}^{\sqrt{R}} 2^{\varpi(m)} \right\}.$$

Since*

$$\sum_{n=1}^{x} 2^{\varpi(n)} = \frac{6}{\pi^2} x \log x + O(x).$$

it follows that

$$N(R) \leqslant \frac{6}{\pi^2} \sqrt{R} . \log R + O(\sqrt{R}).$$

The result of the paper now follows.

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* Landau *Handbuch*, p. 666.

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NOTES AND QUESTIONS.

ASSOCIATE EDITORS:

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Notes and Questions.

On the values of n which make (2n)!/(n+1)! (n+1)! an integer.*

[Mr. Balakram obtained the results of this paper during 1922-23. He did some further work in this connection, which, being mostly numerical, is not given here.—S. D. CHOWLA.]

1. We know that

$$\frac{(2n)!}{n!\ n!}$$

is an integer. Now

$$\frac{(2n)!}{n!\,(n+1)!} = \frac{(2n)!}{n!\ n!} - \frac{(2n)!}{(n+1)!\ (n-1)!}.$$

Hence

$$\frac{(2n)!}{n!\ (n+1)!}$$

also is an integer. But

$$\frac{(2n)!}{(n+1)!\ (n+1)!}$$

is not always an integer. The only cases from $n = 1$ to $n = 100$ in which it is an integer are when

$$n = 5, 14, 27, 41, 44, 65, 76, 90.$$

2. We denote the expression

$$\frac{(2n)!}{(n+1)!\ (n+1)!}$$

by $G(n)$. Our problem† is to find the values of n which make $G(n)$ an integer. I have not found anything like a complete answer to this question, but I prove some interesting results, such as

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* Communicated by S. D. Chowla

† Suggested by Prof Baldev Puri.

Theorem I. *$G(n-1)$ is an integer when n is an even perfect number.*

Theorem II. *$G(n)$ is an integer for an infinity of n's.*

3. **Lemma.** *$G(n-1)$ is an integer if and only if*

$$\sum_{m=1}^{\infty} I\left(\frac{2n-2}{p^m}\right) \geqslant 2\sum_{m=1}^{\infty} I\left(\frac{n}{p^m}\right)$$

for every prime factor p of n.

Here $I(x)$ is the greatest integer contained in x.

Proof. We have

$$G(n-1) = \frac{(2n-2)!}{n!\ n!}.$$

Now, if p^k be the highest power of the prime p, which divides x!

it is well-known that $k = \sum_{m=1}^{\infty} I\left(\frac{x}{p^m}\right)$. Hence:

$$x! = \Pi\, p^{\sum_1^{\infty} I\left(\frac{x}{p^m}\right)}$$

where p runs through all the primes not exceeding x. Hence $G(n-1)$ is an integer if and only if

$$\sum_{m=1}^{\infty} I\left(\frac{2n-2}{p^m}\right) \geqslant 2\sum_{1}^{\infty} I\left(\frac{n}{p^m}\right)$$

for all primes p not exceeding n.

Now if p is not a prime factor of n, then

$$I\left(\frac{2n-2}{p^m}\right) \geqslant 2\,I\left(\frac{n-1}{p^m}\right)$$

$$= 2\,I\left(\frac{n}{p^m}\right).$$

Hence

$$\sum_{m=1}^{\infty} \mathrm{I}\left(\frac{2n-2}{p^m}\right) \geqslant 2\sum_{1}^{\infty} \mathrm{I}\left(\frac{n}{p^m}\right)$$

when p is not a prime factor of n. The lemma is therefore proved.

4. Now let p be a prime such that $2^p - 1$ is also a prime. We know that all even perfect numbers are of the form

$$n = 2^{p-1}(2^p - 1).$$

It is easy to verify that

$$\frac{n!}{2^{n-p}}$$

is odd, and that

$$\frac{(2n-2)!}{2^{2n-2p}}$$

is odd. Hence when

$$n = 2^{p-1}(2^p - 1)$$

where p is a prime, then

$$\frac{(2n-2)!}{n!\,n!} = \frac{a}{b}$$

where a and b are odd integers.

Again if $m = 2^p - 1$ we see that

$$\mathrm{I}\left(\frac{2n-2}{m}\right) + \mathrm{I}\left(\frac{2n-2}{m^2}\right) + \dots = 2^p$$

$$2\left\{\mathrm{I}\left(\frac{n}{m}\right) + \mathrm{I}\left(\frac{n}{m^2}\right) + \dots\right\} = 2^p.$$

From the lemma we now see that

$$\frac{(2n-2)!}{n!\,n!}$$

is an integer if n is an even perfect number, which is Theorem I.

5. We now prove

Theorem III. *If $n = pq$ where p and q are primes and*

$$\tfrac{3}{2}p < q < 2p,$$

then

$$G(n-1) = \frac{(2n-2)!}{n!\,n!}$$

is an integer.

Proof. We have

$$2\sum_{m=1}^{\infty} I\left(\frac{n}{p^m}\right) = 2(q+1)$$

$$\sum_{1}^{\infty} I\left(\frac{2n-2}{p^m}\right) = 2q+2$$

$$2\sum_{1}^{\infty} I\left(\frac{n}{q^m}\right) = 2p$$

$$\sum_{1}^{\infty} I\left(\frac{2n-2}{q^m}\right) = 2p.$$

From these and our lemma, Therem III follows.

6. If $\pi(x)$ is the number of primes not exceeding x it is well-known that

$$\pi(x) \sim \frac{x}{\log x}.$$

Hence to every large prime p, there is at least one other prime q such that

$$\tfrac{3}{2}p < q < 2p.$$

Hence Theorem II is only a special case of Theorem III.

Balakram.

On the least circle enclosing a given set of points in a plane *

1. The following results are fundamental and form the starting point of our investigation :—

(i) The least circle enclosing two points is the circle on the join of the points as diameter.

(ii) The least circle enclosing three points which are the vertices of an obtuse-angled or right-angled triangle is the circle on the greatest side as diameter.

(iii) Of all circles through a vertex of a triangle, enclosing the triangle, the circumcircle is the least, provided the angles at the remaining vertices are acute

For, let ADE be any circle enclosing the triangle ABC, where the angles B and C are acute. Produce BC to cut the circle at D, E. Then $AE > AC$ and $\angle ADE < \angle ABC < \pi/2$; hence

$$\frac{AE}{\sin ADE} > \frac{AC}{\sin ABC}.$$

(iv) The least circle enclosing an acute-angled triangle is its circum-circle.

For, let O be the centre of a circle enclosing the acute-angled triangle ABC and not passing through any of the vertices. Then the circle with O as centre and radius equal to the greatest of the lengths OA, OB, OC also encloses the triangle and is smaller; and the least of such circles is by (iii) the circumcircle.

2. Given a set consisting of a finite number of points in a plane, it is sufficient to consider (for purposes of enclosure) only the outlying points forming a convex polygon enclosing all the other points. We shall

* *Vide* Question No. 11887 by Prof. Sylvester in the ***Educational Times*** which is as follows: "Find the least circle which will contain a given system of points in a plane; also the least sphere which will contain a given system of points in space" An analytical solution is outlined by S V. Ramamurthi, pp. 31-32 of ***Mathematics from the Educational Times*** New Series, Vol. XIX.

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therefore take up next the investigation of the least circle enclosing a given convex polygon $A_1A_2 \ldots A_n$.

Let O be the centre and R the radius of the minimum circle enclosing the polygon.

(i) Consider the lengths OA_1, OA_2, ... OA_n; there must be some one, OA_p say, which is either the greatest or one of the greatest of these lengths. Then OA_p must be equal to R; for, otherwise $OA_p < R$ and the circle centre O and radius OA_p will enclose the polygon and will be less than the circle centre O and radius R: but this latter is the minimum circle. Thus *the minimum circle should pass through at least one vertex, A_p, of the polygon.*

(ii) Let $A_pA'_p$ be the diameter of the minimum circle. Then A'_p may be a vertex of the polygon or not. In the latter case, consider the angles subtended by $A_pA'_p$ at each of the remaining vertices. None of these can be acute, since the circle encloses the polygon. If we erect perpendiculars at A_1, A_2, ... to A_pA_1, A_pA_2, ... these must meet $A_pA'_p$ at points B_1, B_2, ... between A_p and A'_p. Of the points B_1, B_2, ... there must be at least one, say B_q, which is farthest from A_p. If B_q does not coincide with A'_p, the circle on A_pB_q as diameter will enclose all the vertices and will be smaller than the circle on $A_pA'_p$ as diameter which is the minimum circle. Thus B_q coincides with A'_p and so the minimum circle passes through A_q. Hence *the minimum circle should pass through at least two vertices, A_p, A_q, say, of the polygon.*

(iii) Now let $\angle A_pA_uA_q$ $(= \alpha)$ be the least angle subtended by A_pA_q at any of the vertices on one side of A_pA_q and $\angle A_pA_vA_q$ $(= \beta)$ be similarly the least angle subtended on the other side. If a circle through A_p, A_q encloses the polygon, $\alpha + \beta \geqslant \pi$ so that α and β cannot both be acute.

Now, if neither α nor β is acute, the minimum circle must be the circle on A_pA_q as diameter.

If one of these, say α, is acute, then both the circles $A_pA_uA_q$, $A_pA_vA_q$ enclose the polygon and when they do not coincide, the circle $A_pA_uA_q$ is the lesser circle. It is also the least of all the circles through A_p, A_q enclosing the polygon.

3. We shall, in what follows, exclude the cases in which

(*a*) the minimum circle enclosing the polygon is the circle on the line joining two of its vertices as diameter;

(*b*) four vertices of the polygon are concyclic.

Under these suppositions, the results of para. 2 show that the minimum circle enclosing the polygon should pass through three of its vertices. The problem of determining the minimum circle reduces, therefore, to finding the least of the circumcircles (which enclose the polygon) of triangles of the form $A_kA_lA_m$.

A triangle whose vertices are any three vertices of the given polygon we shall refer to as "a triangle of the polygon."

We shall now consider a convex polygon $A_1A_2 \ldots A_n$ of n vertices, subject to the restrictions (*a*) and (*b*) above and we shall investigate some properties of triangles of this polygon whose circumcircles enclose the polygon.

4. At any vertex, A_1 say, of the polygon, there is a definite net of triangles $A_1A_2A_k$, $A_1A_kA_l$, ... $A_1A_rA_n$ $(2 < k < l < \ldots < r < n)$ the circumcircle of each of which encloses the polygon.

For, take the side A_1A_2; let A_k be the vertex at which A_1A_2 subtends the least angle, α_1 say. Let A_l be the vertex, on the side of A_1A_k away from A_2, at which A_1A_k subtends the least angle α_2. Similarly let A_m be the vertex, on the side of A_1A_l away from A_2, A_k, at which A_1A_l subtends the least angle α_3 and so on. This sequence must come to an end, and A_n must be the last vertex thus reached, A_r being the penultimate vertex and $\angle A_1A_nA_r$ $(= \alpha_8)$ being the least angle subtended by A_1A_r on the side of A_1A_r away from A_2, A_k ...

It is evident that this net must consist of at least one triangle $(A_1A_2A_n)$ and may consist of any number up to $n-2$. We shall refer to this net as "the net of triangles belonging to A_1."

The following are some properties of this net:

(i) *The circumcircle of each of the triangles of the net encloses the polygon.*

Denote the angles $A_1A_2A_k$, $A_1A_kA_l$, $A_1A_lA_m$ ... ... $A_1A_rA_n$ by $\beta_1, \beta_2, \beta_3, \ldots \beta_\delta$. Now it is clear that circle $A_1A_2A_k$ encloses the polygon. Hence $\beta_1+\alpha_2>\pi$, and β_1 is the least angle subtended by A_1A_k at any vertex on the same side of A_1A_k as A_2. So the circle $A_1A_kA_l$ encloses the polygon. Hence, again $\beta_2+\alpha_3>\pi$, and β_2 is the least angle subtended by A_1A_l at any vertex on the same side of A_1A_l as A_2, A_k; and so on, for all the triangles of the net.

Thus, we get $\beta_1+\alpha_2>\pi$, $\beta_2+\alpha_3>\pi$, ... $\beta_{\delta-1}+\alpha_\delta>\pi$, and just as $\alpha_2, \alpha_3, \ldots \alpha_\delta$ are the least angles subtended by A_1A_k, A_1A_l, A_1A_r on one side, similarly, $\beta_1, \beta_2, \ldots \beta_{\delta-1}$ are the least angles subtended by them on the other side and β_δ is the least angle subtended by A_1A_n.

The angles α_1 and β_δ are acute and of the pairs of angles (β_1, α_2), (β_2, α_3) ... $(\beta_{\delta-1}, \alpha_\delta)$ one must be acute and the other obtuse. This follows from the inequalities above and the restriction in 3 (*a*). These considerations, coupled with the fact that each of the pairs $(\alpha_1, \beta_1)(\alpha_2, \beta_2) \ldots (\alpha_\delta, \beta_\delta)$ being two of the angles of a triangle of the net, must together be less than π, show that one and only one of these pairs must be both acute. For, suppose β_s $(1 \leqslant s \leqslant \delta)$ to be the first β to become acute so that $\beta_1, \beta_2, \ldots \beta_{s-1}$ are all obtuse Then $\alpha_2, \alpha_3 \ldots \alpha_s$ must all be acute Also α_{s+1} must be obtuse, β_{s+1} acute, α_{s+2} obtuse and so on. Thus every β after β_s is also acute and every α after α_s is obtuse. Hence (α_s, β_s) is the one and only one pair both of which are acute. In every other pair, one must be acute and the other obtuse. If the net comprises only one triangle, it has both its base angles acute. Thus we conclude that

(ii) *In the net of triangles belonging to* A_1, *there is one and only one triangle whose base angles are both acute.*

We shall refer to this triangle as *the triangle associated with the vertex* A_1. It can be easily seen that the circumcircle of the triangle associated with A_1 is the least of the circles passing through A_1 and enclosing the polygon. The circumcircles of the triangles of the net increase, in order, on either side of this minimum circle.

(iii) ***Every triangle (of the polygon) which has one vertex at A_1 and whose circumcircle encloses the polygon, is comprised in the net belonging to A_1.***

For let $A_1A_iA_j$ be a triangle ($j > i$) whose circumcircle encloses the polygon. Then, as before, we can find the sequence of vertices A_h, A_g, ... ($i > h > g$ ...) on the side of A_1A_i away from A_j at which A_1A_i, A_1A_h, ... subtend the least angles respectively. By this process, we must ultimately reach A_2. If A_c be the vertex reached penultimate to A_2, then $A_1A_cA_2$ should be the least angle subtended by A_1A_2 at any of the vertices and therefore A_c must coincide with A_k. Hence, successively the vertices A_c, ... A_g, A_h, A_i, A_j must coincide in order with the vertices A_k, A_l, A_m ... in the net of A_1. Thus $A_1A_iA_j$ is a member of the net of A_1.

(iv) ***Each of the triangles of the net of A_1, other than that which is associated with A_1, may be considered to be associated with that vertex, which contains the obtuse angle.***

5. From the above investigation, especially (iii) and (iv) of para. 4, it is evident that every triangle of the polygon whose circumcircle encloses the polygon, is associated uniquely with one of its vertices, unless the triangle is acute-angled in which case it has to be associated with each of three vertices.

Now all the triangles of the polygon whose circumcircles enclose the polygon cannot be obtuse-angled, for none of their circumcircles can be the minimum circle. Hence at least one of the triangles must be acute-angled. There cannot also be more than one such acute-angled triangle. For, if possible, suppose $A_pA_qA_r$ and $A_xA_yA_z$ are two acute-angled triangles whose circumcircles C_1, C_2 enclose the polygon. Then by para. 1, C_1 is the least circle enclosing $A_pA_qA_r$. Hence $C_2 \nless C_1$, since C_2 also encloses C_1. For a similar reason $C_1 \nless C_2$. So C_1 must be equal to C_2. But if two equal circles intersect, the arc of each intercepted within the other is less than a semicircle and hence any three points on either of these arcs give an obtuse-angled triangle. So the two circles must coincide, *i.e.*, A_p, A_q, A_r, A_x, A_y, A_z lie on a circle which is excepted by the restriction in para. 3 (*b*).

Hence we have the theorem that

There is one and only one acute-angled triangle of the polygon whose circumcircle encloses the polygon. This is the minimum circle enclosing the polygon.

Again of the triangles whose circumcircles enclose the polygon there is one associated with each vertex; each obtuse-angled triangle is associated with only one vertex, *viz.*, the obtuse-angled vertex, and the acute-angled triangle is associated with each of the three vertices. Hence it follows that

There are $n-2$ triangles of the polygon, and no more, having the property that their circumcircles enclose the polygon, with one among them as the minimum circle.

Lastly, it may be noted that *these $n-2$ triangles are non-overlapping and together make up the entire polygon.*

A. A. KRISHNASWAMI AYYANGAR.

Tangential Inversion.

Notation. Let ABC be the triangle of reference and

$$L = lx + my + nz = 0,$$

the equation of a straight line in areal co-ordinates. Then the line

$$L' = \frac{x}{l} + \frac{y}{m} + \frac{z}{n} = 0$$

is defined to be the *tangential inverse* of L with respect to the triangle ABC. This transformation applied to lines is somewhat analogous to isogonal transformation and would appear worthy of study.

1. A line and its inverse are the asymptotes of a circum-conic.

This follows from the fact that the asymptotes of a conic are such that the products of the perpendiculars from any point of the conic on them is constant and this is true in the present case.

2. If L, L′ meet the sides in P, Q, R, P′, Q′, R′ respectively, then by a known property of conics, PP′ and BC have the same mid-point,

This gives us a method of constructing L′ when L is given in position, and incidentally a construction for the centre of the conic when three points on it and one asymptote are given.

3. Through a given point P there are only two lines which are mutually inverse and these are the asymptotes of the circum-conic whose centre is P.

4. A line coincides with its tangential inverse in four cases, *i.e.*, when it is one of the three lines joining the mid.-points of the sides of the triangle or the line at infinity.

5. A line and its inverse are orthogonal, if they are tangents to the Steiner's tricusp of the triangle.

The condition that L, L′ are perpendicular is

$$l(m-n)^2 \cot A + m(n-l)^2 \cot B + n(l-m)^2 \cot C = 0. \quad \ldots \quad (1)$$

This is known to be the tangential equation of Steiner's tricusp. The equation (1) is thus the envelope of the asymptotes of rectangular hyperbolas circumscribing the given triangle. Knowing that the centres of the rectangular hyperbolas lie on the nine-points circle, we deduce that orthogonal tangentially inverse lines intersect on the nine-points circle. It is verified that the curve (1) touches the sides and altitudes of the triangle and is therefore the unique three-cusped hypocycloid or Deltoid touching the six joins of an orthocentric tetrad.*

6. The envelope of pairs of tangential inverse lines intersecting on a line $px + qy + rz = 0$ is a curve of class three.

The envelope is

$$pl(m-n)^2 + qm(n-l)^2 + rn(l-m)^2 = 0. \quad \ldots \quad (2)$$

This is the envelope of the asymptotes of circum-conics having $px + qy + rz = 0$ as the centre-locus, or put differently the envelope of the asymptotes of conics touching the four lines :

$$x\sqrt{p} \pm y\sqrt{q} \pm z\sqrt{r} = 0.$$

7. If L passes through a given point P (ξ, η, ζ), then L′ envelopes the conic

$$\xi . mn + \eta . nl + \zeta . lm = 0. \quad \ldots \quad (3)$$

* *Vide J.I.M.S.*, Vol. XVII (1928), p. 232.

This is an in-conic. If N is its centre, it can b3 shown that, (G being the centroid of the triangle),

$$PG : GN = 2 : 1. \qquad \ldots \quad (4)$$

To each line through P, there is a corresponding tangent to (3) and we deduce the theorem:

"The asymptotes of the circum-conic with centre P are tangents to the in-conic whose centre is N, such that PG : GN = 2 : 1."

Another result is

"If one asymptote of a circum-conic touches the maximum inscribed ellipse, the other passes through the centroid."

8. The g-points* of the lines L, L′ are the images of each other in the centroid.

9. It may be verified that the lines L, L′ are conjugate lines with respect to all self-conjugate parabolas, *i.e.*, parabolas which touch the sides of the medial triangle. Hence when L is perpendicular to L′, they are the axis and the tangent at the vertex of a self-conjugate parabola, the envelope of which is, by para. 5 above, the Steiner's tricusp.

10 The envelope of the asymptotes of conics through four points is a bi-parabola.†

Let
$$lx + my + nz = 0$$
and
$$mn.x + nl.y + lm.z = 0$$

be the asymptotes of a conic through A, B, C and a fourth point D, (f, g, h). The perpendiculars from A, B, C, D on the asymptotes are as

$$l : m : n : \frac{lf + mg + nh}{f + g + h}$$

and

$$mn : nl : lm : \frac{mn.f + nl.g + lm.h}{f + g + h}$$

respectively. As their products are equal

$$lmn = \frac{(lf + mg + nh)(mnf + nlg + lmh)}{(f + g + h)^2} \qquad \ldots \quad (5)$$

* The g-point of a line is the mean centre of its intersections with the sides.

† A bi-parabola is a curve of class three, having the line at infinity as a bi-tangent.

Therefore the envelope can also be written in the form

$$ghl(m-n)^2+hfm\,(n-l)^2+fgn\,(l-m)^2=0. \quad \ldots \quad (6)$$

This is the unique bi-parabola touching the six joins of the four points A, B, C, D.

II. The envelope of pairs of tangentially inverse lines, which include a constant angle, is a pair of Deltoids.*

Let $$\phi\,(l,\,m,\,n)=\Sigma\,a^2\,(l-m)\,(l-n). \quad \ldots \quad (7$$

The angle θ between the pair of inverse lines $(l,\,m,\,n)$ and $(mn,\,nl,\,lm)$ is given by

$$\sec\theta\,.\left[mn\frac{\partial\phi}{\partial l}+nl\frac{\partial\phi}{\partial m}+lm\frac{\partial\phi}{\partial n}\right]=2\sqrt{\phi\,(l,m,n)\,\phi\,(mn,\,nl,\,lm)} \quad \ldots \quad (8)$$

Squaring and reducing we get the envelope to be

$$4\,.\,\Delta\,(m-n)\,(n-l)\,(l-m)=\pm\tan\theta\,[\,\Sigma\,a^3\,(m+n)\,(n-l)\,(l-m)\,].$$

That the curves are Deltoids can be proved by finding their foci, which on verification in a manner indicated in a paper of mine,† are at infinity and which consequently being third class curves must be Deltoids. The curve $\Sigma\,a^2\,(m+n)\,(n-l)\,(l-m)=0$ is the Steiner's tricusp given by (1) above.

N. DURAIRAJAN.

[The correspondence studied in the above note is the general quadratic transformation in the line-plane in which one of the fixed elements is the line at infinity. Some of the main properties of the transformation are given in the *Encyclopädie der Mathematischen Wissenchaften*: *Neuere Driecksgeometrie*: III 1, pages 1211, 1212 where a pair of tangentially inverse lines are referred to as "Seitengegengeraden."—*Ed.*]

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* This is merely another form of the known result that the envelope of the asymptotes of circum-conics, which are similar (and therefore have a constant angle between their asymptote) is a pair of Deltoids. *Vide* Hilton: *Plane Algebraic Curves*, p. 273, Ex 21.

† This Journal, Vol. XVII (1928), p. 224.

Solutions.

Question 197.

(K. J. SANJAÑA, M.A.) :—If $N^{\frac{1}{2}}$, when converted into an infinite continued fraction, has a recurring period of $2k$ partial quotients and if the krth convergent be denoted by C_r, prove that

$$C_{2r} = \tfrac{1}{2}\left\{\frac{N}{C_r} + C_r\right\} \qquad \ldots \quad (1)$$

$$C_{m+n} = \left\{\frac{N}{C_m} + C_n\right\} \div \left\{\frac{C_n}{C_m} + 1\right\} \qquad \ldots \quad (2)$$

where r, m and n are any integers.*

Solution by B. Ramamurthi.

Let us first prove $\quad C_2 = \tfrac{1}{2}\left\{\frac{N}{C_1} + C_1\right\} \qquad \ldots \quad (A)$

With the usual notation, if the continued fraction is

$$\sqrt{N} = a + \frac{1}{a_1 +}\,\frac{1}{a_2 +}\cdots\frac{1}{a_{k-1} +}\,\frac{1}{a_k +}\,\frac{1}{a_{k-1} +}\cdots + \frac{1}{a_1 +}\,\frac{1}{2a_1},$$

we have $\quad \dfrac{p_{k-1}}{p_k} = \dfrac{1}{a_{k-1} +}\,\dfrac{1}{a_{k-2} +}\cdots\dfrac{1}{a_1 +}\,\dfrac{1}{a}.$

and $\quad \dfrac{q_{k-1}}{q_k} = \dfrac{1}{a_{k-1} +}\,\dfrac{1}{a_{k-2} +}\cdots + \dfrac{1}{a_1}.$

Hence $\quad \dfrac{1}{a_{k-1} +}\,\dfrac{1}{a_{k-2} +}\cdots\dfrac{1}{a_1 +}\,\dfrac{1}{a + \sqrt{N}} = \dfrac{p_{k-1} + \sqrt{N}.\,q_{k-1}}{p_k + \sqrt{N}\,q_k}.$

$$\therefore \quad \sqrt{N} = a + \frac{1}{a_1 +}\cdots\frac{1}{a_k +}\,\frac{p_{k-1} + \sqrt{N}\,q_{k-1}}{p_k + \sqrt{N}\,q_k}.$$

$$= \frac{\left\{a_k + \dfrac{p_{k-1} + \sqrt{N}\,q_{k-1}}{p_k + \sqrt{N}\,q_k}\right\} p_k + p_{k-1}}{\left\{a_k + \dfrac{p_{k-1} + \sqrt{N}\,q_{k-1}}{p_k + \sqrt{N}\,q_k}\right\} q_k + q_{k-1}}.$$

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* The results of this question are generalisations of the results of Questions 57 [*J.I.M.S.*, Vol., I, pp. 106–8] and 189 [Vol, II, p. 121].—Ed

Putting $\quad a_k p_k + p_{k-1} = p_{k+1}, \; a_k q_k + q_{k-1} = q_{k+1}$

and equating rational parts, we get

$$N\,(q_k q_{k+1} + q_k q_{k-1}) = p_k p_{k+1} + p_k p_{k-1}. \qquad \ldots \quad \text{(B)}$$

Now $\quad C_2 = a + \dfrac{1}{a_1 +} \cdots + \dfrac{1}{a_{k-1} +} \; \dfrac{1}{a_k + q_{k-1}/q_k}$

$$= \frac{p_{k+1}\, q_k + p_k\, q_{k-1}}{q_{k+1}\, q_k + q_k \cdot q_{k-1}}.$$

$$= \frac{2p_{k+1}\, q_k + 2p_k\, q_{k-1}}{2\,(q_{k+1}\, q_k + q_k\, q_{k-1})}$$

$$= \frac{p_{k+1}\, q_k + (-1)^{k+1} + p_k\, q_{k+1} + p_k q_{k-1} + (-1)^k + p_{k-1}\, q_k}{2\,(q_{k+1}\, q_k + q_k \cdot q_{k-1})}$$

$$= \frac{p_k}{2q_k} + \frac{p_{k+1} + p_{k-1}}{2\,(q_{k+1} + q_{k-1})}$$

$$= \frac{p_k}{2q_k} + \frac{N.q_k}{2p_k} \quad \text{by (B)}$$

$$= \frac{N + C_1^2}{2C_1}.$$

Next we shall prove the formula (2), *viz.*

$$C_{m+n} = \frac{N + C_m \cdot C_n}{C_m + C_n}$$

in the case where m or n is an even integer. Let m be even $(= 2x)$. Then

$$C_{2x+n} = \frac{(a + C_n)\; p_{2xk} + p'_{2xk}}{(a + C_n)\; q_{2xk} + q'_{2xk}}$$

$$= \frac{C_n\, p_{2xk} + Nq_{2xk}}{C_n\, q_{2xk} + p_{2xk}}$$

from the well-known relations

$$p_{2xk} = aq_{2xk} + q'_{2xk}$$

$$Nq_{2xk} = ap_{2xk} + p'_{2xk}$$

which are obtained from

$$\sqrt{N} = \frac{(a + \sqrt{N})\, p_{2xk} + p'_{2xk}}{(a + \sqrt{N})\, q_{2xk} + q'_{2xk}}$$

Hence

$$C_{2x+n} = \frac{N + C_n C_{2x}}{C_{2x} + C_n} \quad \ldots \text{(C)}$$

whatever be n.

From (C) writing $n = 1$,

$$C_{2x+1} = \frac{N + C_1 C_{2x}}{C_{2x} + C_1}$$

whence

$$C_{2x} = \frac{N - C_1 C_{2x+1}}{C_{2x+1} - C_1}. \quad \ldots \text{(D)}$$

Also writing $n = 2$,

$$C_{2x+2} = \frac{N + C_2 . C_{2x}}{C_{2x} + C_2}$$

which on using (A) and (D) gives

$$C_{2x+2} = \frac{N + C_1 C_{2x+1}}{C_{2x+1} + C_1}. \quad \ldots \text{(E)}$$

Now we can prove the formula (2) when m and n are both odd, equal to $2x + 1$, $2y + 1$ say. For,

$$C_{2x+2y+2} = \frac{N + C_{2x} . C_{2y+2}}{C_{2x} + C_{2y+2}} \text{ by (C)}$$

Substituting for C_{2x} in terms of C_{2x+1} from (D) and for C_{2y+2} in terms of C_{2y+1} from (E) and simplifying we get

$$C_{2x+2y+2} = \frac{N + C_{2x+1} C_{2y+1}}{C_{2x+1} + C_{2y+1}}.$$

Thus formula (2) is universally true. Writing $m = n = r$, we get (1).

Question 722.

(S. RAMANUJAN):—Solve completely

$$x^2 = a + y,\ y^2 = a + z,\ z^2 = a + u,\ u^2 = a + x;$$

and deduce that, if $x = \sqrt{[5 + \sqrt{\{5 + \sqrt{(5 - \sqrt{(5 + x)})}\}}]}$,

then $x = \frac{1}{2}\{2 + \sqrt{5} + \sqrt{(15 - 6\sqrt{5})}\}$;

and that, if $x = \sqrt{[5 + \sqrt{\{5 - \sqrt{(5 - \sqrt{(5 + x)})}\}}]}$,

then

$$x = \tfrac{1}{4}[\sqrt{5} - 2 + \sqrt{(13 - 4\sqrt{5})} + \sqrt{\{50 + 12\sqrt{5} - 2\sqrt{(65 - 20\sqrt{5})}\}}].$$

*Solution by Prof. G. N. Watson.**

When we express y, z, u and x successively in terms of x, we get

$$y = x^2 - a,\quad z = (x^2 - a)^2 - a,$$

$$u = \{(x^2 - a)^2 - a\}^2 - a,\quad x = [\{(x^2 - a)^2 - a\}^2 - a]^2 - a,$$

and so x satisfies the equation of the sixteenth degree

$$x^{16} - 8ax^{14} + (28a^2 - 4a)\,x^{12} - (56a^3 - 24a^2)\,x^{10}$$
$$+ (70a^4 - 60a^3 + 6a^2 - 2a)\,x^8 - (56a^5 - 80a^4 + 24a^3 - 8a^2)\,x^6$$
$$+ (28a^6 - 60a^5 + 36a^4 - 16a^3 + 4a^2)x^4 - (8a^7 - 24a^6 + 24a^5 - 16a^4 + 8a^3)x^2$$
$$- x + (a^8 - 4a^7 + 6a^6 - 6a^5 + 5a^4 - 2a^3 + a^2 - a) = 0;$$

and similarly y, z and u satisfy the same equation.

If two alternate unknowns of the sequence $x, y, z, u, x, y, \ldots\ldots$ (say x and z) are equal, then it follows from the equations

$$y = x^2 - a,\quad u = z^2 - a,$$

that y and u are also equal, in which case the unknowns satisfy the quartic $\qquad x = (x^2 - a)^2 - a,$

so that we expect $x^4 - 2ax^2 - x + (a^2 - a)$ to be a factor of the left-

* Since completing this solution, my attention has been drawn to Mr. M. Bhimasena Rao's paper on "Cyclic Equations" in the *J. I. M. S.*, Vol. XVI, pp. 139—154.

hand side of the equation of the sixteenth degree; this is actually the case, the remaining factor giving rise to the duodecimic equation

$$x^{12} - 6ax^{10} + x^9 + (15a^2 - 3a)x^8 - 4ax^7 - (20a^3 - 12a^2 - 1)x^6$$
$$+ (6a^2 - 2a)x^5 + (15a^4 - 18a^3 + 3a^2 - 4a)x^4 - (4a^3 - 4a^2 - 1)x^3$$
$$- (6a^5 - 12a^4 + 6a^3 - 5a^2 + a)x^2 + (a^4 - 2a^3 + a^2 - 2a)x$$
$$+ (a^6 - 3a^5 + 3a^4 - 3a^3 + 2a^2 + 1) = 0.$$

We first solve the quartic by observing that it is satisfied if consecutive unknowns of the sequence $x, y, z, u, x, y, \ldots\ldots$ (say x and y) are equal; for, if $y = x$ and $y = x^2 - a$, then $z = y^2 - a = x^2 - a = x$, and similarly $u = x$ also. Hence we expect $x^2 - x - a$ to be a factor of

$$x^4 - 2ax^2 - x + (a^2 - a),$$

and the remaining quadratic factor is, in fact, $x^2 + x - (a - 1)$.

We thus have obtained the following solutions of the given system:

(I) $x = y = z = u = \frac{1}{2}\{1 + \sqrt{(4a + 1)}\}$.

(II) $x = y = z = u = \frac{1}{2}\{1 - \sqrt{(4a + 1)}\}$.

(III) $x = z = -\frac{1}{2}\{1 \pm \sqrt{(4a - 3)}\}, \; y = u = -\frac{1}{2}\{1 \mp \sqrt{(4a - 3)}\}$.

We have now considered all possible ways in which two of the unknowns can be equal. We now have to determine the solutions for which all the unknowns are unequal; these solutions are the roots of the duodecimic equation, and we shall find that these roots can be grouped into three sets of four, each set forming a set of values of x, y, z and u.

Returning to the original system, choose new unknowns, P and Q, defined by the equations

$$x + z = 2P + 2Q, \quad y + u = 2P - 2Q.$$

Then, from the given equations,

$$x^2 + z^2 = 2a + y + u = 2a + 2P - 2Q,$$

so that
$$2xz = (x + z)^2 - (x^2 + z^2)$$
$$= 4(P + Q)^2 - 2a - 2P + 2Q.$$

Hence x and z are the roots of the quadratic in X

$$X^2 - 2(P + Q)X + 2(P + Q)^2 - a - P + Q = 0.$$

Similarly y and u are the roots of the quadratic in X

$$X^2 - 2(P - Q)X + 2(P - Q)^2 - a - P - Q = 0.$$

From the former of these, it follows that $x + a$ and $z + a$ are the roots of the equation

$$X^2 - 2(P + Q + a)X + (P + Q + a)^2 + (P + Q)^2 - a - P + Q = 0,$$

while, from the latter of them, y^2 and u^2 are the roots of the equation

$$[X + 2(P - Q)^2 - a - P - Q]^2 - 4X(P - Q)^2 = 0.$$

From the given system we see that the last two quadratics must be identical; the co-efficients of X^2 and of X are 1 and $-2(P + Q + a)$ in both of them, and, when we compare their constant terms, we find that

$$4(P - Q)^4 - 4(P - Q)^2(P + Q + a) - (P + Q)^2 + a + P - Q = 0,$$

Similarly, by comparing the quadratic whose roots are $y + a$ and $u + a$ with the quadratic whose roots are x^2 and z^2, we find that

$$4(P + Q)^4 - 4(P + Q)^2(P - Q + a) - (P - Q)^2 + a + P + Q = 0,$$

an equation obviously obtainable by changing the sign of Q in the previous equation.

Subtracting these equations, we get

$$Q\,[(16P + 4)Q^2 + 16P^3 - 4P^2 + (2 - 8a)P + 1] = 0.$$

The possibility of Q being zero may be rejected, since it would make the quadratic for x and z identical with the quadratic for y and u, so that we merely re-obtain the solution when the four unknowns are equal. Hence we must have

$$(16P + 4)Q^2 + 16P^3 - 4P^2 + (2 - 8a)P + 1 = 0.$$

Again, from the given system

$$(x - z)(x + z) = y - u, \quad (y - u)(y + u) = z - x;$$

multiplying these and rejecting the non-vanishing factors $x-z$ and $y-u$, we have

$$(x+z)(y+u) = -1,$$

and hence at once $$Q^2 = P^2 + \tfrac{1}{4}.$$

Substituting this value of Q^2 in the preceding equation connecting P and Q, we obtain an equation in P which becomes on simplification

$$16P^3 + (3-4a)P + 1 = 0.$$

If P is taken to be each of the three roots in turn of this irreducible cubic, the roots of the pairs of quadratics

$$X^2 - 2PX + 4P^2 + \tfrac{1}{2} - a - P \mp (2X - 4P - 1)\sqrt{(P^2 + \tfrac{1}{4})} = 0$$

give the three sets of four roots of the duodecimic equation. The pairs of quadratics may be combined into the quartics

$$X^4 - 4PX^3 + (8P^2 - 2P - 2a)X^2 + (8P^2 + 4aP + 2P + 1)X$$
$$- 8aP^2 - 2aP + a^2 - a + 1 = 0.$$

I have verified that the product of the three quartics of this type gives the duodecimic equation.

In each of the numerical examples a has been taken equal to 5, so that the equation for P is

$$16P^3 - 17P + 1 = 0,$$

with roots 1, $-\tfrac{1}{4}(2 \pm \sqrt{5})$.

Taking $P = 1$ and $Q = +\tfrac{1}{2}\sqrt{5}$, the equation with roots x and z is

$$X^2 - (2+\sqrt{5})X + \tfrac{1}{2}(5\sqrt{5} - 3) = 0,$$

while the equation with roots y and u is

$$X^2 + (\sqrt{5} - 2)X - \tfrac{1}{2}(5\sqrt{5} + 3) = 0.$$

Taking x to be $\tfrac{1}{2}\{2 + \sqrt{5} + \sqrt{(15 - 6\sqrt{5})}\}$, we have

$x > z > 0$ so that $\quad y = x^2 - 5 > z^2 - 5 > u,$

and so, since the roots of the quadratic for y and u have opposite signs,

$$y > 0 \text{ and } u < 0.$$

Hence $$x = +\sqrt{[5+y]} = \sqrt{[5+\sqrt{\{5+z\}}]}$$
$$= \sqrt{[5+\sqrt{\{5+\sqrt{(5+u)}\}}]}$$

and $$u = -\sqrt{(5+x)},$$

so that $$x = \sqrt{[5+\sqrt{\{5+\sqrt{(5-\sqrt{(5+x)})}\}}]}.$$

The second example can be discussed in the same manner, taking

$$P = \tfrac{1}{4}(\sqrt{5}-2),\quad Q = +\tfrac{1}{4}\sqrt{(13-4\sqrt{5})};$$

if x is then taken to be the *positive* root of the quadratic

$$X^2 - \tfrac{1}{2}\{\sqrt{5}-2+\sqrt{(13-4\sqrt{5})}\}X - \tfrac{1}{4}\{7+5\sqrt{5}+\sqrt{(13-4\sqrt{5})}\} =$$

it is found that $y > 0$, $z < 0$, $u < 0$, and hence that

$$x = \sqrt{[5+\sqrt{\{5-\sqrt{(5-\sqrt{(5+x)})}\}}]}.$$

Question 1422.

(A. Narasinga Rao):—Show that if three tangents to a tricusp hypocycloid are concurrent, then the three tangents perpendicular to these form a triangle whose pedal lines touch the tricusp.

Solution by N. Sundaram Aiyar.

Let b be the radius of the rolling circle. Then the tangent at

$$b(2\cos\theta + \cos 2\theta),\ b(2\sin\theta - \sin 2\theta)$$

is $$x\sin\frac{\theta}{2} + y\cos\frac{\theta}{2} = b\sin\frac{3\theta}{2}. \quad \ldots \quad (1)$$

The condition for this to pass through (h, k), leads to a cubic equation in $\cot\frac{\theta}{2}$. If the three roots are $\cot\frac{\theta_1}{2}$, $\cot\frac{\theta_2}{2}$, $\cot\frac{\theta_3}{2}$, we see easily that

$$\Sigma\cot\frac{\theta_2}{2}\cot\frac{\theta_3}{2} = 1 \text{ or } \theta_1+\theta_2+\theta_3 = 2n\pi,\ (n, \text{ integral})$$

as the relation between the parameters of the points of contact.

Also $$x\sin\frac{\theta+\pi}{2} + y\cos\frac{\theta+\pi}{2} = b\sin\frac{3}{2}(\theta+\pi) \quad \ldots \quad (2)$$

is the tangent perpendicular to (1), *i.e.*, $\theta + \pi$ is the parameter of the point of contact of the tangent perpendicular to (1).

If $\theta_1 + \theta_2 + \theta_3$ is an even multiple of π, $(\theta_1 + \pi) + (\theta_2 + \pi) + \theta_3$ and two similar expressions are also even multiples of π, that is, the tangents at the corresponding points are concurrent.

This shows that the three tangents perpendicular to the three tangents concurrent at S (h, k) intersect two by two on the latter, *i.e.*, S is the orthocentre of the triangle formed by the latter tangents. The pedal lines of this Δ ABC, say, touch the tricusp hypocycloid whose centre is the centre of the nine-point circle, and the radius of the rolling circle of which is the radius of the nine-point circle.*

Also the orthoptic locus of the tricusp hypocycloid, got by squaring and adding the two sides of (1) and (2) is seen to be

$$x^2 + y^2 = b^2,$$

a circle whose centre is the centre of the tricusp and whose radius is that of the rolling circle.

The envelope of the pedal lines of ABC touches the sides and altitudes of Δ ABC, as they are only particular positions of the pedal lines. As the altitudes of Δ ABC are the original tangents from S, the feet of the altitudes of Δ ABC are on the orthoptic loci of the two tricusps. As the orthoptic locus is a circle, determined by three points, the two tricusps have the same orthoptic locus. Hence the centres of the two tricusps are the same, as well as the radii of the rolling circles in the two cases. In other words, the two tricusps are the same. Hence the pedal lines of the Δ ABC touch the original tricusp.

Also solved by B. Ramamurti.

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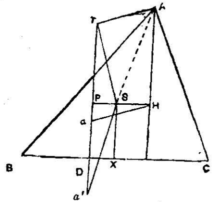
* *Vide* Basset: *Cubic and Quartic Curves, Art.* 325, 330.

Question 1446.

(V. RAMASWAMI AIYAR):—ABC and $\alpha\beta\gamma$ are two directly similar triangles, such that the orthocentre of the former is the circum-centre of the latter. Show that the reflexions of α, β, γ in BC, CA, AB lie on a circle whose centre is the circum-centre of ABC.

Solution by T. R. Raghava Sastri and M. V. Seshadri.

Let H be the orthocentre of ABC and circumcentre of $\alpha\beta\gamma$, S the circumcentre of ABC, X the mid-point of BC, and α', the reflexion of α. Draw AT parallel to $H\alpha$ to meet $\alpha\alpha'$ in T. Join ST and $S\alpha'$. Draw SP parallel to BC meeting $\alpha\alpha'$ in P.



Since $HAT\alpha$ is a parallelogram, $HA = \alpha T$

But $HA = 2\,SX.$

Hence $T\alpha' = T\alpha + \alpha\alpha' = HA + \alpha\alpha'$

$= 2\,SX + 2\,\alpha D.$

$= 2\,(PD + D\alpha') = 2\,P\alpha'.$

Hence P is the mid-point of $T\alpha'$. $\therefore\ ST = S\alpha'$.

Also the triangle SAT is fixed, since SA is the circum-radius R of ABC, $AT = H\alpha = r$, the circum-radius of $\alpha\beta\gamma$ and the angle SAT is equal to θ, the angle between any two corresponding lines. Hence ST is fixed and is equal to

$$\sqrt{R^2 + r^2 - 2Rr\cos\theta}.$$

Similarly $S\beta' = S\gamma' = S\alpha'$.

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1552. (A. A. KRISHNASWAMI AIYANGAR) :—Find the right-circular cylinder of minimum cross-section enclosing a given set of points in space.

1553. (C. N. SRINIVASIENGAR) :—Prove the following properties of Steiner's quartic surface :—

(1) The parabolic curve consists of three parabolas and constitutes the envelope of the asymptotic lines.

(2) There exists a quartic curve on the surface the indicatrix at any point of which is a plane cubic.

1554. (C. N. SRINIVASIENGAR) :—

(1) Prove that the surface defined by the arc-element $ds^2 = 2f(p)\,dp\,dq$ is a developable having two systems of plane lines of curvature.

(2) Prove that the surface defined by the values of the fundamental magnitudes

$$E = 0, \quad F = f(p), \quad G = 0,$$

$$L = \frac{a^2}{b}\{f(p)\}^2, \quad M = af(p), \quad N = b$$

where a and b are any constants, is a right circular cylinder whose radius of cross-section is equal to $(2a)^{-1}$.

1555. (S. SIVASANKARANARAYANA PILLAI) :—When m is a given positive integer, prove that $m^a + 1$ has got no prime factors of the form $4rm - 1$.

1556. (S. SIVASANKARANARAYANA PILLAI) :—

If $$\phi_r(n) = \Sigma\, \phi(d_1)\, \phi(d_2) \ldots \ldots \phi(d_r)$$

where $$d_1 d_2 \ldots \ldots d_r = n,$$

then $$\phi_r(n) = \phi(n) \prod_{p/n} \left\{ {}_rC_1 + {}_rC_2 \times {}_{\alpha-1}C_1 \times \left(1 - \frac{1}{p}\right) + \ldots \ldots + {}_rC_t \times {}_{\alpha-1}C_{t-1} \times \left(1 - \frac{1}{p}\right)^{t-1} + \ldots \ldots \right\}$$

where α is the greatest integer such that p^α is a divisor of n.

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LIST OF JOURNALS RECEIVED IN THE LIBRARY

during the months of August and September 1929.

1 American Journal of Mathematics, **51**, 3. (3 copies).

2 American Mathematical Monthly, **36**, 6.

3 Annals of Mathematics, **30**, 3.

4 Astrophysical Journal, **69**, 4, 5.

5 Bulletin of the Calcutta Mathematical Society, **21**, 2.

6 Bulletin des Sciences Mathematiques, **53**, June issue.

7 Half-yearly Journal of the Mysore University, **3**, 1.

8 Japanese Journal of Mathematics, **6**, 2.

9 Mathematical Gazette, **14**, 202. (3 copies).

10 Mathematische Annalen, **101**, 5, **102**, 1.

11 Messenger of Mathematics, **58**, 1, 2 (2 copies each), 3 (3 copies).

12 Nieuw Archief Voor Wiskunde, Vol. 6.

13 Philosophical Magazine, **8**, 48, 49.

14 Philosophical Transactions of the Royal Society, **228**, pp. 331—376.

15 Proceedings of the Edinburgh Mathematical Society, **1**, 4.

16 Proceedings of the London Mathematical Society, **29**, 6, 7 (2 copies each).

17 Proceedings of the Royal Society of London, **124**, 795*a*, **125**, 796 (2 copies), **125**, 797.

18 Publication de la Faculte des Sciences, No. 87.

19 Rendiconti del Circolo Mathematico de Palermo, **53**, 2.

20 Tohoku Mathematical Journal, **31**, 1, 2.

21 Transactions of the Royal Society of South Africa, **18**, 1, 2.

Books.

1 Physical Development of Negro-white Hybrids in Jamaica (British West Indies) by Morris Steggerda.

2 Analytic Functions of Hypercomplex Variables by P. W. Ketchum.

3 Attitudes as factors of college success by Marion Eugene Herriott.

4 Examination Papers of the Bombay University, 1928.

5 Hand-book of the University of Bombay, 1929-1930.

6 Calendar of the University of Madras for 1929-30.

Numbers in black type refer to the volumes, and those in ordinary type to the numbers of the issues.

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